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The splitting of branes on orientifold planes*

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Abstract: Continuing the study in hep-th/0004092, we investigate a non-trivial string dynamical process related to orientifold planes, i.e., the splitting of physical NS-branes and D$(p+2)$-branes on orientifold Op-planes. Creation or annihilation of physical Dp-branes usually accompanies the splitting process. In the particular case $p=4$, we use Seiberg-Witten curves as an independent method to check the results.

Keywords: D-branes, Brane Dynamics in Gauge Theories.

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Contents

1. Introduction

The brane setup [1] (for a review, see [2]) has been proved to be a very useful tool to relate string theory to field theory. On one hand, we can use known results in string theory to get non-trivial new results in field theory. For instance, mirror symmetry in three dimensions [3] can be easily studied by brane setup [1], [4]–[8]. On the other hand, field

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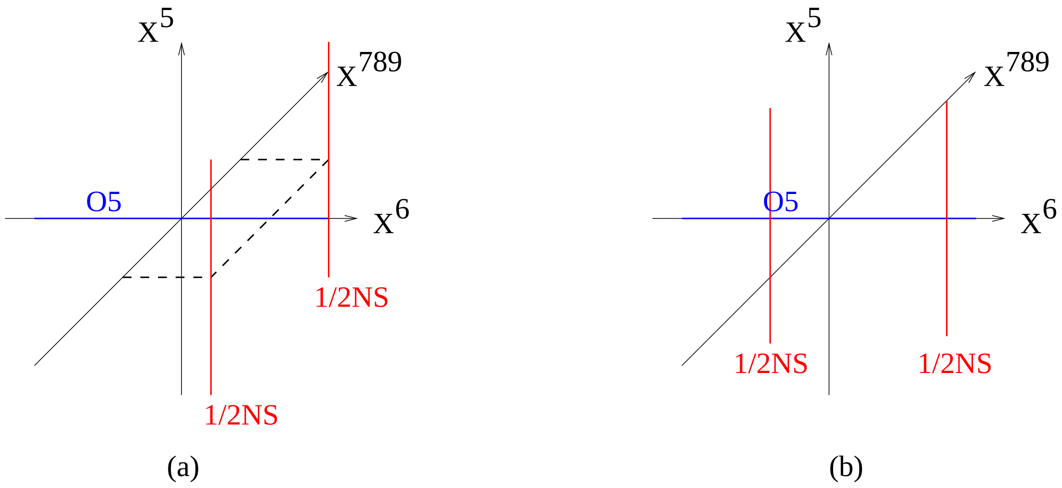

Figure 1: (a) 1/2NS-brane and its image have the same X^6 coordinate. (b) After splitting, they have different X^6 coordinates.

theory considerations can be used to predict non-trivial dynamics in string theory. One of the most known examples is the generation or annihilation of a D3-brane when a D5-brane crosses an NS-brane [1]. This prediction and its generalizations were demonstrated later from various points of view [9]–[16].

In this note, we want to study one prediction about string dynamics which was found in [8] using consistency arguments within field theory. In [8], in order to construct mirror pairs by $O3$-planes, two of us made a conjecture on the splitting of a physical brane (NS-brane and D5-brane) on the $O3$-planes. The general picture of splitting is given in figure 1. Initially, a 1/2NS-brane with non-vanishing X^{789} is separated from its image (see part (a)). Then, we move them towards the orientifold plane at $X^{789} = 0$. When they touch the orientifold, they are on top of each other. Since now each piece is mirror to itself, they can move along the X^6-direction independently (see part (b)). This process is called "splitting".

The results of splitting in [8][1] are as follows. A physical D5-brane splits without the generation of branes on the $O3^+, O3^-, \widetilde{O3^+}$ planes, while a physical D3-brane is generated on the $\widetilde{O3^-}$-plane. The splitting of a physical NS-brane can be derived from the D5-brane case by S-duality. The NS-brane will split freely, namely without the generation of branes, on $\widetilde{O3^-}, O3^-, \widetilde{O3^+}$ planes, while a physical D3-brane is generated on the $O3^+$ plane.

We got the above results indirectly by using field theory arguments. It would be very interesting to gather more evidence to support them, or even derive them directly. Furthermore, it would be very nice to discuss how NS-branes and D$(p+2)$-branes split on Op-planes for arbitrary p. It turns out that the splitting of D$(p+2)$-branes on Op-planes is universal for $p \leq 6$, namely the conclusions we have derived for the case of D5-branes on O3-planes hold for all p. However, the splitting of NS-branes on Op-planes is more complex and there is no universal behavior. Each case has to be discussed separately. The results

[1]For simplicity, we will only state the results of the splitting of physical branes which have no net D3-branes ending on them. In order to obtain the more general case, we can use the same trick given in [8].

can be summarized in the following table:

Type	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
Op^+	0	0	1	1	≤ 2	$M-4$
$\widetilde{Op}^+$	0	0	0	0	≤ 1	$M-7/2$
Op^-	0	0	0	-1	≤ -2	$M+4$
$\widetilde{Op}^-$	0	0	0	-1	≤ -2	$M+7/2$

(1.1)

In the table above, 1 means that a Dp-brane is generated while -1 means that a Dp-brane is annihilated. For the annihilation to be possible, we need enough Dp-branes on top of the orientifold before the splitting. The M for the $p=6$ case is the cosmological constant which we will discuss in more detail later. For $p=5$ case, the ≤ 2 means that any number of branes which is less than or equal to two is allowed (we use the convention that a negative number means annihilation).

The above conclusions are derived by several consistency arguments. In the special case of $p=4$, we can check the result by studying the Coulomb branch of $\mathcal{N}=2$ symplectic and orthogonal gauge theories realized in this brane setup. In particular, we will analyze the corresponding Seiberg-Witten curves.

The plan of the paper is as follows. In section 2, we derive the general splitting rules for both D-branes and NS-branes. In section 3, we explain how to check these rules in the specific case of the NS-brane splitting on $O4$-planes by Seiberg-Witten curves. Then, we introduce the curves for $\mathcal{N}=2$ theories with a single symplectic or orthogonal gauge group. In section 4, we write down the curves for a product of symplectic and orthogonal gauge groups and then study the NS-brane splitting.

2. The general picture of the splitting

In this section, we will give the general picture of the splitting of physical D-branes and NS-branes on Op-planes. We will argue that the splitting of D-branes on orientifolds is universal, while the splitting of NS-branes is not.

The plan for this section is as follows. In section 2.1, we study the universal splitting of D-branes. In section 2.2, we observe that for NS-branes, the splitting is not universal so that we need to discuss each case separately. In section 2.3, we use the minimum energy argument to derive the splitting of NS-branes on Op-planes with $p \leq 4$. In section 2.4 and 2.5, we use proper consistency conditions, i.e., the cosmological constant argument for $p=6$ and charge conservation for $p=5$, to reach the desired results.

2.1 The universal splitting of D-branes

First, let us discuss the splitting of a physical D$(p+2)$-brane on $Op^{\pm}, \widetilde{Op}^{\pm}$-planes for $p \leq 6$.[2] The D$(p+2)$-brane is extended along the $X^{012\ldots(p-1)}$ and X^{789} directions, while

[2]Recently the authors of [22] discussed the existence of $\widetilde{Op}^-$ for $p=6,7,8$. They showed that it is indeed possible for $p=6$, in the presence of a non-zero cosmological constant, but it is not allowed for $p=7,8$. If $\widetilde{O6}^-$ exists, there should be $\widetilde{O6}^+$ as well, since crossing a 1/2NS-brane changes minus charged planes to plus charged ones [23, 24].

the Op-plane is extended along the $X^{012\ldots(p-1)}$ and X^6 directions.[3] There are actually more than four kinds of $O1$-planes, but we will not discuss all of them in this paper and refer the reader to [17, 18, 19]. We will give several arguments to show that the splitting of a physical D$(p+2)$-brane on Op-planes is universal: the D$(p+2)$-brane will split freely on $Op^+, Op^-, \widetilde{Op}^+$ planes, while a physical Dp-brane is created on the $\widetilde{Op}^-$-plane.[4]

The first argument, which was employed in [8], involves the Higgs mechanism. Let us consider the familiar brane setup given by two 1/2NS-branes, k physical Dp-branes and n physical D$(p+2)$-branes which preserves 8 supercharges. The Dp-branes are parallel to the Op-plane and the 1/2NS-branes are extended along the X^{012345} directions. Depending on the type of Op-planes, we have different p-dimensional gauge theories: Sp(k) for Op^+, Sp$'(k)$ for $\widetilde{Op}^+$, SO$(2k)$ for Op^- and SO$(2k+1)$ for $\widetilde{Op}^-$. All of them have n hypermultiplets, except for Sp$'(k)$ gauge theory which has two extra half-hypermultiplets. We can Higgs the above theories down to, for example, Sp$(k-1)$, Sp$(k-1)$, SO$(2k-1)$ and SO$(2k)$ by breaking one physical Dp-brane between the NS-branes and the D$(p+2)$-branes. Now we need to match the degrees of freedom before and after higgsing. From the field theory point of view, when we Higgs the gauge theory, one vector multiplet always eats one hypermultiplet. This is true in any dimension, because there are 8 supercharges.[5] So, after higgsing, we get the lower rank gauge theories, some hypermultiplets in the fundamental representation and some hypermultiplets in the singlet representation. From the brane setup point of view, the higgsing simply corresponds to the breaking of Dp-branes between 1/2NS-branes and 1/2D$(p+2)$-branes. Matching[6] the degrees of freedom that can be read from the brane setup with field theory, we find the results given at the end of the last paragraph.

To make the above argument more transparent, let us consider one example, i.e., the splitting of a D$(p+2)$-brane on the $\widetilde{Op}^-$-plane. The brane setup is given in figure 2. The gauge theory is SO$(2k+1)$ with N hypermultiplets (see part (a)). After splitting the D$(p+2)$-brane, a Dp-brane is generated as in part (b). Then we move the 1/2D$(p+2)$-branes across the two 1/2NS-branes. From part (c), we see that the remaining theory is SO$(2k)$ with $N-1$ hypermultiplets and N singlets.[7] Notice that the Dp-brane generated by the splitting of the D$(p+2)$-brane does not contribute any vector multiplet. Now, let us count the various degrees of freedom. Before higgsing, we have $N(2k+1) = 2kN + N$ degrees of freedom. After higgsing, we have $(N-1)2k + N = 2kN + N - 2k$ only. However, taking into account the $k(2k+1) - k(2k-1) = 2k$ degrees of freedom which have been eaten by the vector multiplets, the two numbers match exactly, $2kN + N - 2k + 2k = 2kN + N$. The same argument can be applied to other cases.

[3] We focus on the case where the brane setup preserves 8 supercharges.

[4] Further observations supporting such a universal behaviour will be presented when we discuss the splitting of NS-branes.

[5] This fact can be easily seen by noticing that one massive vector multiplet has the same number of degrees of freedom as one massless vector multiplet and one massless hypermultiplet.

[6] When we do the matching, we need to know the generalized supersymmetric configuration. This is given in appendix A. For more details, the reader may refer to [8] where one explicit example is given.

[7] One easy way to see that there are N singlets is that $N-1$ D$(p+2)$-branes will cut the generated Dp-brane into N pieces.

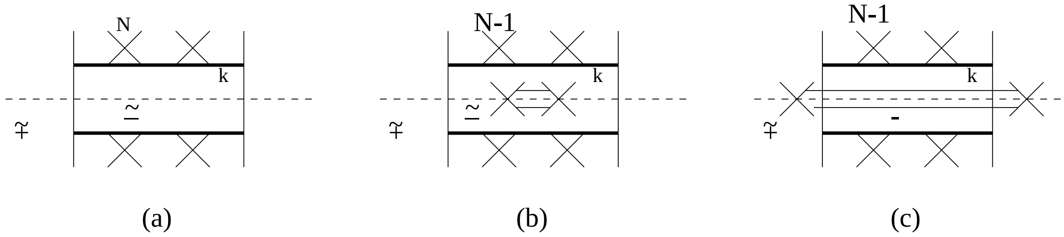

Figure 2: (a) The brane setup of the gauge theory. (b) The splitting of a D$(p+2)$-brane on the $\widetilde{Op}^{-}$-plane with the generation of one Dp-brane. (c) The final theory after higgsing.

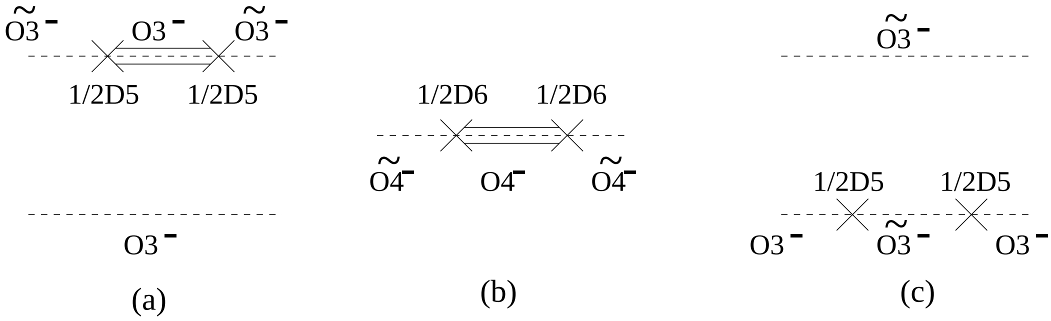

Figure 3: Part (a) shows that a D3-brane is generated when a D5-brane splits on the upper $\widetilde{O3}^{-}$-plane. Part (b) is T-dual to Part (a). (c) Instead of splitting on the upper plane, we split the D5-brane on the lower $O3^-$ plane. No D3-branes appear and after T-duality we recover Part (b) again.

The second argument is based on T-duality. Let us give an example, namely the derivation of the splitting of a D6-brane on $\widetilde{O4}^{-}$, to illustrate the idea. We start from the known result of the splitting of a D5-brane on $\widetilde{O3}^{-}$. First, we compactify the X^3 direction and get two fixed $O3$-planes which are shown in part (a) of figure 3. It is important to notice that the two 1/2D5-branes intersect only the upper $O3$-plane. Notice also that, in order to get $\widetilde{O4}^{-}$ after T-duality, we set the lower $O3$-plane to be $O3^-$. After T-duality, $\widetilde{O3}^{-}$ and $O3^-$ combine to give $\widetilde{O4}^{-}$, while $O3^-$ and $O3^-$ combine to give an $O4^-$, as in part (b). From part (b), we can read off that a physical D4-brane is generated when a physical D6-brane splits on $\widetilde{O4}^{-}$. We can also reach part (b) by splitting one D5-brane on the lower $O3^-$-plane, as shown in part (c). Since two $\widetilde{O3}^{-}$-planes yield one $O4^-$ with one physical D4-brane, we obtain the same result.

The T-duality argument can be easily generalized to other cases. One will find the universal behavior we claimed before. For instance, we can compactify the X^2 direction instead of the X^3 direction to find the splitting of a D4-brane on $O2$-planes. In this case, it is important to recall the result of [19] that, after T-duality, the $O3$-plane will be divided into two $O2$-planes, but the 1/2D4-branes will intersect only one $O2$-plane.

There are other universal behaviours, like the bending of the branes and the charge difference of the orientifold planes on the two sides of a 1/2D$(p+2)$-brane. They will be addressed in next subsection when we discuss the splitting of NS-branes.

All of this evidence supports the claim that the splitting of D$(p+2)$-branes on Op-planes is universal.

2.2 The non-universal behaviour of the NS-brane splitting

Now let us discuss the splitting of physical NS-branes on Op-planes. One may think that we can solve this problem by T-duality, but there are actually several observations that indicate the failure of naive T-duality. For example, it is well known that a D4-brane is created after an NS-brane crosses a D6-brane. But after T-duality, there is no brane creation after a Kaluza-Klein monopole crosses a D5-brane, as shown in [20].

Let us recall that the NS-branes are extended along the X^{012345} directions and that the Op-plane is extended along the $X^{012\cdots(p-1)}$ and X^6 directions.

First of all, let us consider the bending of an NS-brane due to the intersection with Dp-branes. From Witten's work [21], such a bending can be found approximately by solving

$$\bigtriangledown^2 x^6 = q\delta^{(6-p)}(r-r_0)\,, \tag{2.1}$$

where function $x^6(r)$ represents the bending of the NS-brane along the X^6 direction, q is the D-brane charge deposited at the intersection points r_0 and $\bigtriangledown^2$ is the laplacian operator in the NS-brane worldvolume coordinates. Far away from the Op-plane, the bending of x^6 goes like r^{p-4}. So, when $p \leq 3$, the bending caused by the intersection will disappear at large distance, while when $p = 4, 5$ the bending will propagate to infinity. This transition from $p = 3$ to $p = 4$ is the first sign that we should be careful while applying T-duality.

Let us compare the above bending of NS-branes with the one of D$(p+2)$-branes. Since a D$(p+2)$-brane always has three non-compact transverse dimensions relative to the Dp-brane, its bending goes like r^{-1} and disappears at infinity, which does not depend on p. This difference between the non-universal bending of NS-branes and the universal bending of D$(p+2)$-branes indicates why T-duality is applicable to the D-brane system but not to the NS-brane system.

The second observation is related to the charge difference between the two Op-planes on the two sides of the intersection. On one hand, at the intersection with a 1/2NS-brane, the charge difference is $2\times 2^{p-5} = 2^{p-4}$ for the $Op^{\pm}$ pair and $2^{p-5} - (\frac{1}{2} - 2^{p-5}) = 2^{p-4} - \frac{1}{2}$ for the $\widetilde{Op}^{\pm}$ pair, both depending on p. On the other hand, at the intersection with a 1/2D$(p+2)$-brane, the charge difference is always 1/2 for the $Op^-, \widetilde{Op}^-$ pair and zero for $Op^+, \widetilde{Op}^+$ pair, no matter what p is.

The third observation concerns the different behaviour of NS-branes and D-branes under T-duality. If we T-dualize one dimension down, we get two fixed $O(p-1)$-planes. A 1/2NS-brane will intersect both fixed planes, while a 1/2D$(p+1)$-brane (the T-dual of a 1/2D$(p+2)$-brane) will intersect only one of these two fixed planes. These different intersection properties have very important consequences. For example, if T-duality is correct, the D3-brane generated on the $O3^+$-plane by splitting the NS-brane should be divided into two equal parts on both $O2^+$-planes, i.e. there is a 1/2D2-brane generated when an NS-brane splits on an $O2^+$-plane, which is not allowed. If we went further, we would come up with the conclusion that there is a quarter physical D1-brane generated when an NS-brane splits on an $O1^+$-plane. This is obviously wrong. In the D-brane case, we do not have this problem because the D-branes will intersect with and split on only one of these two fixed planes, hence we do not need to divide the effect into two parts.

On these grounds, it is clear we should rely on other methods to discuss the splitting of NS-branes on Op-planes. In the case of $O3$-planes, we can apply S-duality. For $p \leq 4$, we can use a minimum energy argument. For $p = 5$, we need to analyze the relative (p, q)-web. For $p = 6$, the configuration has to be consistent with RR charge conservation in a background with a possibly non-vanishing cosmological constant. The details of these methods are given in the following subsections.

2.3 The minimum energy argument

There are two sources of energy costs when we try to split a physical brane into two half branes (two $1/2$D$(p+2)$-branes or $1/2$NS-branes) on the Op-plane. The first one is the energy it takes to bend a $1/2$D$(p+2)$-brane or $1/2$NS-brane. The bending energy is proportional to the absolute value of the Dp-brane charge difference between the two sides. The second one is due to the creation (which costs energy) or annihilation (which produces energy) of Dp-branes, parallel to the Op-plane, between the two $1/2$-branes. The minimum configuration is the balance between these two factors. For $p \leq 4$, since the $1/2$NS-brane has more transverse directions than the Dp-branes, the energy cost from the first source dominates. For $p = 5, 6$ the second source becomes important so we need to be more careful. However, in this subsection, we focus on the $p \leq 4$ case only.

2.3.1 The D$(p+2)$-brane case

To check that our minimum energy argument is right, we apply it to the splitting of D$(p+2)$-branes. Since D$(p+2)$-branes have two transverse directions more than Dp-branes, the first source is more important. The charge difference between the two sides of the split $1/2$D$(p+2)$-brane is given by the following table

Type	Δq for D(p+2)
Op^+	N
$\widetilde{Op}^+$	N
Op^-	$1/2 + N$
$\widetilde{Op}^-$	$-1/2 + N$

where N is the number of Dp-branes generated in the middle when we split the D$(p+2)$-brane. Note that there is no dependence on p, signalling a universal behaviour for D$(p+2)$-branes. Thus we see that, in order to minimize the energy cost, we need to choose $N = 0$ in the first two cases, $N = 0, -1$ in the third case and $N = 0, 1$ in the last case.

We need a complementary argument to fix the ambiguity in the last two cases. This is provided by field theory as we discussed above. Namely, we need to match the Higgs branch before and after the higgsing mechanism.

The above discussion is applied to all $p \leq 6$, and in particular also to the $\widetilde{O6}^-$-plane, whose existence was argued by [22]. Note that, upon crossing a $1/2$NS-brane, the $\widetilde{O6}^-$-plane becomes a $\widetilde{O6}^+$-plane [23, 24], which indicates the existence of the $\widetilde{O6}^+$-plane. Furthermore, since $O6^\pm$-planes can exist in a background with vanishing cosmological constant, they can also exist with an arbitrary *integer* cosmological constant. This is due

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to the fact that, by crossing an arbitrary number of physical D8-planes, the $O6^{\pm}$-planes remain such while the cosmological constant jumps by an integer. Conversely, since we have found that a physical D8-brane can split into two 1/2D8-branes, we see that the $\widetilde{O6}^{\pm}$-planes can exist in a background with arbitrary *half-integer* cosmological constant [22]. This observation will be important later when we discuss the splitting of NS-branes.

2.3.2 The NS-brane case

Now we calculate the bending energy of a 1/2NS-brane, which for $p \leq 4$ is proportional to the Dp-brane charge difference between the two sides, which is given by

Type	Δq for NS
Op^+	$-2^{p-4} + N$
y$\widetilde{Op^+}$	$-2^{p-4} + \frac{1}{2} + N$
Op^-	$2^{p-4} + N$
$\widetilde{Op^-}$	$-\frac{1}{2} + 2^{p-4} + N$

where N is the number of Dp-branes generated in the middle when we split the NS-brane. Note that the result depends on p, giving evidence of non-universal behavior. The number N corresponding to the minimum energy cost is listed below

Type	$p = 1$	$p = 2$	$p = 3$	$p = 4$
Op^+	0	0	$0, 1$	1
$\widetilde{Op^+}$	0	0	0	$0, 1$
Op^-	0	0	$0, -1$	-1
$\widetilde{Op^-}$	0	0	0	$0, -1$

(2.2)

Except for $p = 1, 2$, the minimum energy requirement does not fix the number uniquely. We need to discuss each case separately. For $p = 3$, S-duality tells us that $N = 1$ for $O3^+$ and $N = 0$ for $O3^-$.

For $p = 4$, we need to take into account that the bending is like $\ln r$, so that when two 1/2NS-branes bend towards each other, they will finally meet and combine to give a physical NS-brane which is not bent. To see it clearly, let us discuss the example of $\widetilde{O4^+}$. If we choose $N = 1$ for $\widetilde{O4^+}$, the distance between these two 1/2NS-branes will become infinite as we move away from the intersection point with the $O4$-plane (see part (a) of figure 4). Such a bending will cost an infinite amount of energy, and we can rule it out. On the other hand, if we choose $N = 0$ for $\widetilde{O4^+}$, these two 1/2NS-branes will meet each other (see part (b) figure 4) and combine to give an NS-brane. This configuration definitely costs less energy, so we should choose $N = 0$. By the same argument, we have that $N = -1$ for the case $\widetilde{O4^-}$.

2.4 The splitting of NS-branes on $O6$-planes

When we consider the NS/D6-brane system, we need to take the cosmological constant M into account [25, 26]. In this background, RR charge conservation requires that

$$N_L - N_R = M\ ,$$

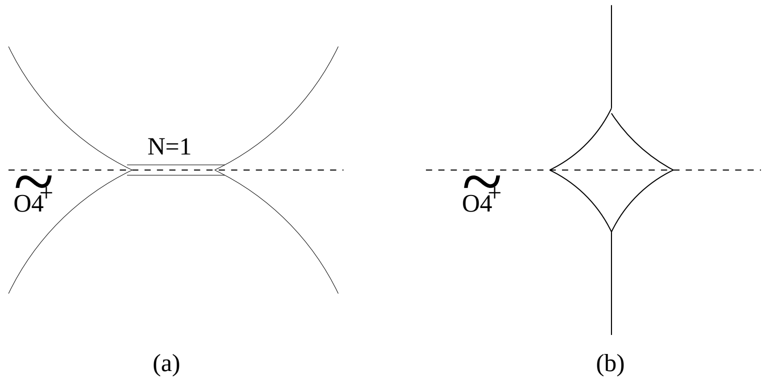

Figure 4: The bending of 1/2NS-branes after splitting on the $\widetilde{O4}^+$-plane for case (a) $N = 1$ and (b) $N = 0$.

where N_L is the number of D6-branes ending on the left hand side of the NS-brane and N_R is the number of D6-branes ending on the right hand side. In the presence of an orientifold, the same formula holds for a 1/2NS-brane with the understanding that N_L, N_R are the D6-brane charges.

The general framework for our discussion is depicted in figure 5. In part (a), we have N_L and N_R 1/2D6-branes ending on the two sides of two 1/2NS-branes and the consistency condition reads

$$\frac{1}{2}(N_L - N_R) = M \,.$$

After the splitting, we have $2N_L$ 1/2D6-branes ending on the left, $2N_R$ 1/2D6-branes ending on the right and $2N$ 1/2D6-branes in the middle (see part (b)). The consistency conditions become

$$[(N_L + q_1) - (N + q_2)] = M \,, \qquad [(N + q_2) - (N_R + q_1)] = M \,,$$

where q_1, q_2 are the charges of the left (or right) and middle orientifold planes respectively. Notice that these two conditions imply that $1/2(N_L - N_R) = M$, which is consistent with part (a). Now we can solve these two equations and get

$$N_L - N = M + q_2 - q_1 \,.$$

Putting different q_1, q_2, we can summarize the results in the following table

q_1 charge	$O6^+$	$O6^-$	$\widetilde{O6}^+$	$\widetilde{O6}^-$
N	$N_L - (M - 4)$	$N_L - (M + 4)$	$N_L - (M - 7/2)$	$N_L - (M + 7/2)$

Note that since N_L, N and N_R are integers, the cosmological constant M has to be an integer in the $O6^{\pm}$ case and half-integer in the $\widetilde{O6}^{\pm}$ case for the splitting to be possible. This condition on M is perfectly consistent with our observation at the end of section 2.1.

2.5 The splitting of NS-branes on $O5$-planes

In this case, things become more interesting. We not only have NS-branes or D5-branes, but also general (p, q)-branes with p, q coprime [27, 28]. Charge conservation at the intersection

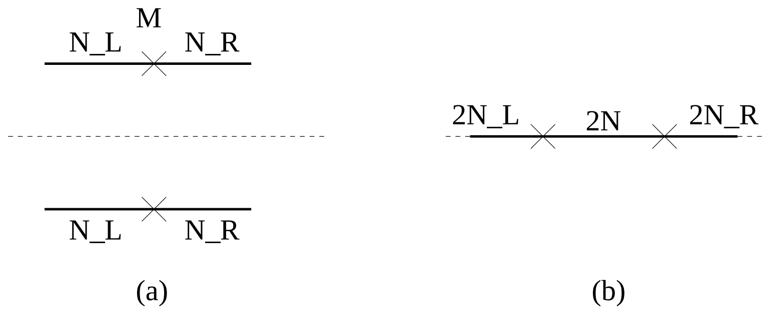

Figure 5: The splitting of an NS-brane on $O6$-planes with cosmological constant M. (a) Before the splitting, we have $1/2(N_L - N_R) = M$. (b) After the splitting, N physical (or $2N$ half) D6-branes are generated.

point becomes the necessary condition for the configuration to be consistent. Supersymmetry further requires the orientation of branes to be fixed by their (p, q)-charges and the string coupling constant g_s.

Let us recall some facts about the general (p, q)-web. Define

$$\tau = \frac{\xi}{2\pi} + \frac{i}{\lambda}. \tag{2.3}$$

The tension of a (p, q)-brane is

$$T_{p,q} = |p + \tau q| T_{D5} \tag{2.4}$$

(from this we see that $(1, 0)$ is a D5-brane and $(0, 1)$ is an NS-brane. This is the convention used in the paper). To preserve a quarter of the original supersymmetry, the orientation of the (p, q)-brane is given by

$$\Delta x + i\Delta y || p + \tau q\,. \tag{2.5}$$

When we include $O5$-planes along the x-direction, to be consistent with the Z_2 orientifold action, τ must be a pure imaginary number, namely $\xi = 0$. If we use this condition, we will have that

$$T_{p,q} = T_{D5}\sqrt{p^2 + |\tau q|^2} \tag{2.6}$$

with the orientation

$$\frac{\Delta x}{\Delta y} = \frac{p}{-i\tau q}\,. \tag{2.7}$$

Now we can start the calculation of the energy cost. The general setup is given in figure 6. When $p_l < p_m + n$, the brane bending is like in part (a) which will cost an infinite amount of energy. This leaves us the condition that $p_l \geq p_m + n$. For $p_l = p_m + n$ with integer n, the brane does not bend and the energy cost is

$$2lnT_{D5} + 2l(p_m - p_l)T_{D5} = 0\,,$$

where the first term is the contribution from the generated D5-branes (remember that negative n means annihilation) and the second term is the contribution from the change of orientifold plane. Next, consider the case $p_l > p_m + n$ as in part (b) of figure 6. Charge

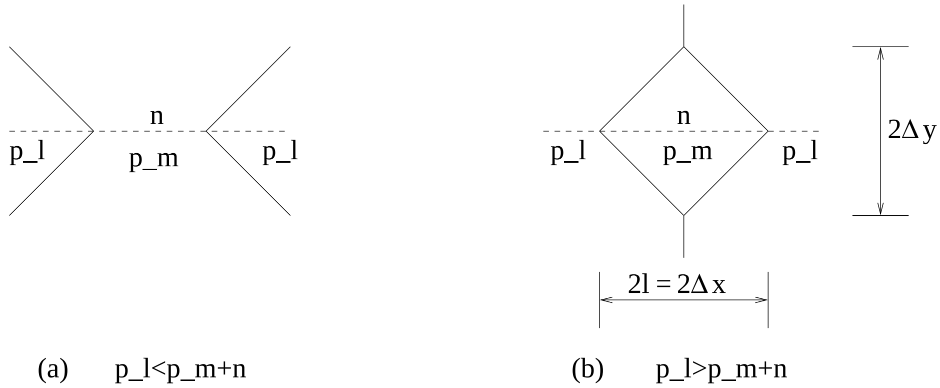

Figure 6: The setup of NS-brane splitting on $O5$-planes, where p_l, p_m are the charges of the orientifolds on the l.h.s. (the same as r.h.s.) and in the middle and n is the number of D5-branes generated in the splitting process. (a) The bending when $p_l < p_m + n$. (b) The bending when $p_l > p_m + n$. In this case, the 1/1-branes meet and combine to become one physical NS-brane again.

conservation tells us that the bent branes are $((p_l - (p_m + n))/2, \pm 1/2)$-branes. From it we get

$$\begin{aligned} |\Delta y| &= \left|\frac{\tau l}{p_l - (p_m + n)}\right|, \\ \text{length} &= \sqrt{l^2 + (\Delta y)^2} = l\sqrt{1 + \left(\frac{|\tau|}{p_l - p_m - n}\right)^2}, \\ E_{\text{bending}} &= 4T_{p,q}\text{length} = lT_{D5}2(p_l - p_m - n)\left[1 + \left(\frac{|\tau|}{p_l - p_m - n}\right)^2\right]. \end{aligned}$$

Now the change of energy is

$$\begin{aligned} \Delta E &= 2lnT_{D5} + 2l(p_m - p_l)T_{D5} + E_{\text{bending}} - 2T_{D5}|\tau|\frac{|\tau| l}{p_l - p_m - n} \\ &= 0\,, \end{aligned}$$

where in the first line, the third term (the bending energy) minus the fourth term (the original non-bending part of the NS-brane) give the energy cost for the bending.

From this calculation we see that as long as $p_l \geq p_m + n$, any n is a good choice since all of them cost the same energy (*zero*). It seems to be a little surprising. However, notice that we can relate configurations of different n by a smooth motion with *zero* energy cost, so that above results make perfect sense. Now inserting the proper values of p_l, p_m, we can find the results shown in the table in the introduction. For example, putting $p_l = +1$ ($O5^+$) and $p_l = -1$ ($O5^-$) we get $n \leq 2$.

There are several points we want to discuss. First, we notice that the change of total energy after splitting on $O5$-planes is *zero*. Does this hold for other Op-planes? It does not seem so. In general situations, the total energy of the system does not change before and after the splitting, but sometimes it does. For example, when an NS-brane splits on an $O3^-$-plane, there is no brane created or annihilated, but the 1/2NS-brane does bend. In other words, in the splitting process we do put energy into the system.

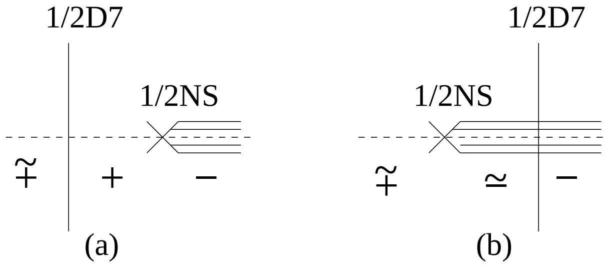

Figure 7: The proof that a 1/2NS-brane does exist on $\widetilde{O5}^{\pm}$-planes. Starting from part (a), which we know is an allowed configuration, by moving the 1/2D7-brane to the right, we reach part (b), where the 1/2NS-brane intersects the $\widetilde{O5}^{\pm}$-plane.

The change of total energy may cause some confusion because it seems that during the whole process the system is supersymmetric and the motion is smooth. In fact, if two parallel branes are static, they do preserve half the amount of initial supersymmetry, but if they move relatively to each other, all supersymmetries are broken. The splitting process, no matter how slow it is, does break supersymmetry, so we cannot use the supersymmetry argument to require that the energy is the same before and after splitting. To make this point more clear, let us recall the following brane setup given in [1]: N D3-branes ending on two parallel NS-branes. If we keep the positions of the D3-branes on the NS-branes invariant and push these two NS-branes away from each other, the bending of the NS-branes will be same, but the length of the D3-branes will increase, which means that the total energy of the brane system does increase.

The second point we want to discuss is that for the $\widetilde{O5}^{\pm}$ cases, we have D-brane charge $(p_l - p_m - n)/2 = \pm 3/4 - n/2$ for the bending brane if we demand the charge conservation condition at the intersection point. The appearance of a quarter charge D-brane is a non-trivial prediction by splitting process. However, the legitimacy of a quarter charge D-brane is not very clear to us at this moment. We have several arguments to support its existence, but we still cannot draw a definite conclusion from them. First, if we do not allow for a quarter charge D-brane, we would violate charge conservation at the intersection point and the energy change before and after splitting would not be zero. The second point is that the D-brane with quarter charge is not complete. It sits at $X^{789} = 0$, like the orientifold plane, and must be accompanied by its image. If we draw them in real space (not the covering space in figure 6), they are on top of each other. So, when we calculate the D5-brane charge, we must sum them together and we get a half-integer charge which would not violate the Dirac quantization condition. In comparison, a 1/2D5-brane and its image can have nonzero X^{789}, so when we draw them in real space, they are not on top of each other and the D-brane charge is not the sum. The third point is that branes with quarter charge and their images can never be on top of the orientifold because, if they were, they would change the type of orientifold and cause an inconsistency. The last point is that a 1/2NS-brane does exist on $\widetilde{O5}^{\pm}$-planes. This can be seen from the following brane construction given in figure 7. In part (a), there are two physical D5-branes ending on the r.h.s. of a 1/2NS-brane. Then, we move the 1/2D7-brane to the right and reach part (b). Since part (a) exists, part (b) must exist too.

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3. Analysis of $p = 4$ by Seiberg-Witten curves

We get the splitting rules in general situations by using various arguments. However, it would be very satisfactory if we had some more direct way to derive them. Although we do not know how to do this for general p, in the case of $O4$-planes, we can check the above results by studying the Coulomb branch of symplectic and orthogonal gauge theories arising on the worldvolume of D4-branes stretched between the 1/2NS-branes.

The general idea is the following. Let us start from the brane setup with N physical D4-branes on top of $O4$-planes and some physical NS-branes above the $O4$-planes (see, for example, part (a) of figure 8). Then, we move the NS-branes to touch the $O4$-planes and split them into 1/2NS-branes, where we assume this is allowed. Next, we split the D4-branes and let them end on the 1/2NS-branes. After these operations, we get various gauge theories between the 1/2NS-branes with rank N' and N. As we know N' can differ from N. As a matter of fact, this is exactly the brane configuration considered in [29] to derive the Seiberg-Witten curves for the Coulomb branch of $\mathcal{N} = 2$ symplectic and orthogonal gauge theories via M-theory, generalizing the work of Witten for the Coulomb branch of $\mathcal{N} = 2$ SU(N) gauge theories [21] (see part (b) of figure 8).

If the above splitting process is correct, we can reverse it. Namely, we can move two 1/2NS-branes towards each other and combine them into one physical NS-brane which can then leave the $O4$-plane.

The low-energy physics in the Coulomb branch is encoded in a polynomial with non trivial dependence on the moduli and scales of the various theories, the Seiberg-Witten curve. If the reverse process is possible, the Seiberg-Witten curve undergoes a specific factorization. In the following, we are going to study the conditions for this transition to happen.

We have to remark that this analysis via the Coulomb branch is not general, in the sense that the gauge theories between the 1/2NS-branes have to be asymptotically free or conformal. This imposes certain conditions on the number of D4-branes ending on the various 1/2NS-branes. The condition has a simple geometrical picture. Since the inverse Yang-Mills coupling, $1/g^2_{\rm YM}$, is proportional to the distance between the 1/2NS-branes, in the AF case the distance between them increases as we move away from the $O4$-plane and the D4-branes, whereas the distance remains invariant in the conformal case [21, 29].

3.1 The general setup

Before delving into a detailed discussion in the next section, let us set up the general framework to investigate the splitting of the NS-brane.

We are going to consider only four 1/2NS-branes with D4-branes stretched between them, as in part (c) of figure 8. In particular, there are no D4-branes on the l.h.s. of A2 nor on the r.h.s. of C1. This configuration corresponds to a product of three gauge theories with a certain number of flavors for each factor: the type of $O4$-plane between two 1/2NS-branes dictates the gauge group on the D4-branes worldvolume. Recall that after crossing a 1/2NS-brane the minus charged $O4$-planes are changed into plus charged $O4$-planes and vice versa: for example, $O4^+$ turns into $O4^-$ and $\widetilde{O4}^-$ turns into $\widetilde{O4}^+$ [23, 24]. In the

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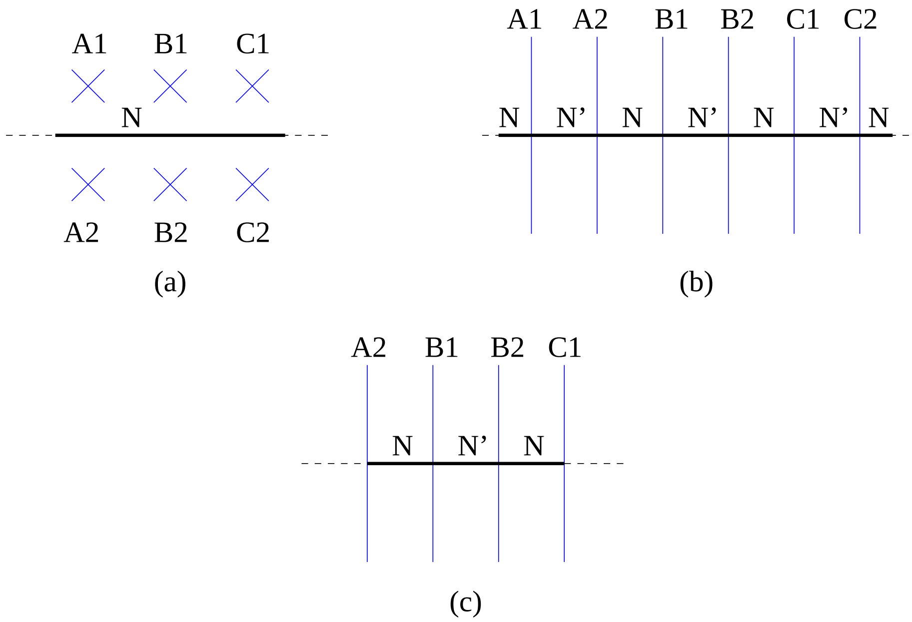

Figure 8: (a) Three physical NS-branes before the splitting. (a) The gauge field theories constructed by the splitting of NS-branes and D4-branes. (c) Our framework of gauge field theories.

following table we list the various gauge group factors from left to right and the number of flavors for each factor depending on the type of $O4$-plane between A2 and B1,

Type	Gauge groups	Number of Flavors
$O4^+$	$\mathrm{Sp}(N) \times \mathrm{SO}(2N') \times \mathrm{Sp}(N)$	$(N', 2N, N')$
$\widetilde{O4}^+$	$\mathrm{Sp}'(N) \times \mathrm{SO}(2N'+1) \times \mathrm{Sp}'(N)$	$(N'+1, 2N, N'+1)$
$O4^-$	$\mathrm{SO}(2N) \times \mathrm{Sp}(N') \times \mathrm{SO}(2N)$	$(N', 2N, N')$
$\widetilde{O4}^-$	$\mathrm{SO}(2N+1) \times \mathrm{Sp}'(N') \times \mathrm{SO}(2N+1)$	$(N', 2N+1, N')$

(3.1)

A careful analysis of the Seiberg-Witten curve will tell us if one can move the two middle 1/2NS-branes on top of each other by properly adjusting the gauge theory moduli and scales. Conversely, if this is possible, a physical NS-brane can be split on the corresponding $O4$-plane.

3.2 The Seiberg-Witten curve of a single gauge group with flavors

The results for the $\mathcal{N} = 2$ SU(2) gauge theory obtained in the celebrated papers [30, 31] have since been generalized to other gauge groups:[8] to SU(n) groups in [32]–[35], to SO($2k$) in [36, 37], to SO($2k+1$) in [37, 38] and to Sp(k) in [39, 40]. In this paper, we will follow the conventions of [39].

3.3 The Seiberg-Witten curves for SO and Sp groups

The Seiberg-Witten curves for Sp(r), SO($2r$) and SO($2r+1$) are given in [39]. The coeffi-

[8] We list only part of the references and apologize to those who are not cited here.

cients of the β-functions are

$$b_{\mathrm{Sp}} = 2r + 2 - N_f\,, \qquad b_{\mathrm{SO}} = N_c - 2 - N_f\,.$$

So when $N_f = 2r+2$ for $\mathrm{Sp}(r)$, $N_f = 2r-2$ for $\mathrm{SO}(2r)$ and $N_f = 2r-1$ for $\mathrm{SO}(2r+1)$, the theories are conformal. It is understood that the following results apply to asymptotically free theories only.

The corresponding curves are summarized in the table

$\mathrm{Sp}(r)$	$xy^2 = (x\prod_{a=1}^{r}(x-\phi_a^2) + g(\tau)\prod_{j=1}^{2r+2} m_j)^2 - g^2(\tau)\prod_{j=1}^{2r+2}(x-m_j^2)$
$\mathrm{SO}(2r)$	$y^2 = x\prod_{a=1}^{r}(x-\phi_a^2)^2 + 4f(\tau)x^3\prod_{j=1}^{2r-2}(x-m_j^2)$
$\mathrm{SO}(2r+1)$	$y^2 = x\prod_{a=1}^{r}(x-\phi_a^2)^2 + 4f(\tau)x^2\prod_{j=1}^{2r-1}(x-m_j^2)$

(3.2)

where

$$g(\tau) = \frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)+\vartheta_4^4(\tau)}\,, \qquad f(\tau) = \frac{\vartheta_2^4(\tau)\vartheta_4^4(\tau)}{\left(\vartheta_2^4(\tau)-\vartheta_4^4(\tau)\right)^2}\,, \tag{3.3}$$

and

$$\tau = \frac{\theta}{\pi} + i\frac{8\pi}{g^2}\,.$$

As we explain in appendix B, following [35] and [39], $f(\tau+1)$ and $g(\tau+1)$ actually yield physically equivalent curves. Matching the brane picture will fix the modular function: we will encounter this feature when we discuss the splitting of NS-branes. Note that sending τ to $\tau+1$ amounts to shifting θ by π.

The Jacobi theta functions are defined by

$$\begin{aligned}
\vartheta_2(\tau) &= \sum_{n\in\mathbb{Z}} q^{(n+\frac{1}{2})^2}\,, & \vartheta_2^4 &= 16q + O(q^3)\,,\\
\vartheta_3(\tau) &= \sum_{n\in\mathbb{Z}} q^{n^2}, & \vartheta_3^4 &= 1 + 8q + O(q^2)\,,\\
\vartheta_4(\tau) &= \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}\,, & \vartheta_4^4 &= 1 - 8q + O(q^2)\,,
\end{aligned} \tag{3.4}$$

$$q \equiv e^{i\pi\tau} = e^{i\pi\left(\frac{\theta}{\pi}+i\frac{8\pi}{g^2}\right)} = e^{i\theta}e^{-8\pi^2/g^2}$$

and they satisfy the Jacobi identity

$$\vartheta_2^4(\tau) - \vartheta_3^4(\tau) + \vartheta_4^4(\tau) = 0\,.$$

Furthermore, in the limit $q \to 0$,

$$g(\tau) \to 8q\,, \qquad f(\tau) \to 16q\,.$$

To get the curve for the asymptotically free theories with N_f flavors, for example $N_f < 2r+2$ for the $\mathrm{Sp}(r)$ group, we take $2r+2-N_f$ masses to be equal to M and then send M to ∞ while keeping $\Lambda^{2r+2-N_f} = \Lambda^{b_{\rm Sp}} = qM^{2r+2-N_f}$ finite[9] (so $q \to 0$ in such limit). By such an operation we get the curve

$\mathrm{Sp}(r)$	$xy^2 = \left(x\prod_{a=1}^{r}(x-\phi_a^2) + 8\Lambda^{b_{\rm Sp}}\prod_{j=1}^{N_f} m_j\right)^2 - (-1)^{b_{\rm Sp}}(8\Lambda^{b_{\rm Sp}})^2\prod_{j=1}^{N_f}(x-m_j^2)$
$\mathrm{SO}(2r)$	$y^2 = x\prod_{a=1}^{r}(x-\phi_a^2)^2 + (-1)^{b_{\rm SO}}64\Lambda^{2b_{\rm SO}}x^3\prod_{j=1}^{N_f}(x-m_j^2)$
$\mathrm{SO}(2r+1)$	$y^2 = x\prod_{a=1}^{r}(x-\phi_a^2)^2 + (-1)^{b_{\rm SO}}64\Lambda^{2b_{\rm SO}}x^2\prod_{j=1}^{N_f}(x-m_j^2)$

(3.5)

3.4 The corresponding curves from M-theory

The corresponding curves of the above gauge groups, i.e. $\mathrm{Sp}(r)$, $\mathrm{SO}(2r)$ and $\mathrm{SO}(2r+1)$ are derived via M-theory in [29]. For later purposes, we are interested in finding the explicit dependence of the various curve parameters on the dynamical scales.

The curves of single gauge groups in [29] can be summarized as

$\mathrm{Sp}(r)$	$\prod_{i=1}^{N_f^L}(v^2-m_i^2)t^2 + e(v^2B(v^2)-c)t + f\prod_{j=1}^{N_f^R}(v^2-m_j^2) = 0$
$\mathrm{SO}(2r)$	$v^2t^2\prod_{i=1}^{N_f^L}(v^2-m_i^2) + eB(v^2)t + hv^2\prod_{j=1}^{N_f^R}(v^2-m_j^2) = 0$
$\mathrm{SO}(2r+1)$	$vt^2\prod_{i=1}^{N_f^L}(v^2-m_i^2) + eB(v^2)t + hv\prod_{j=1}^{N_f^R}(v^2-m_j^2) = 0$

(3.6)

where $B(v^2)$ is a polynomial in v^2 with degree r and e, c ,f, h are constants which may be functions of the flavour masses m_i and the dynamical scale Λ. Furthermore, the constant c is fixed by requiring the curve to have a double root at $v^2 = 0$ [29].

Comparing table (3.6) with (3.5), it is easy to find the correspondence

group	non-conformal	conformal
$\mathrm{Sp}(r)$	$B(v^2) = \prod_{a=1}^{r}(v^2-\phi_a^2)$	$B(v^2) = \prod_{a=1}^{r}(v^2-\phi_a^2)$
	$c = -8\Lambda^{b_{\rm Sp}}\prod_{j=1}^{N_f} m_j$	$c = -g(\tau)\prod_{i=1}^{2r+2} m_i$
	$4f/e^2 = (-1)^{b_{\rm Sp}}(8\Lambda)^{b_{\rm Sp}}$	$4f/e^2 = g^2(\tau)$
$\mathrm{SO}(2r)$	$B^2(v^2) = \prod_{a=1}^{r}(v^2-\phi_a^2)^2$	$B^2(v^2) = \prod_{a=1}^{r}(v^2-\phi_a^2)^2$
	$4h/e^2 = -(-1)^{b_{\rm SO}}64\Lambda^{2b_{\rm SO}}$	$4h/e^2 = -4f(\tau)$
$\mathrm{SO}(2r+1)$	$B^2(v^2) = \prod_{a=1}^{r}(v^2-\phi_a^2)^2$	$B^2(v^2) = \prod_{a=1}^{r}(v^2-\phi_a^2)^2$
	$4h/e^2 = -(-1)^{b_{\rm SO}}64\Lambda^{2b_{\rm SO}}$	$4h/e^2 = -4f(\tau)$

(3.7)

where we list the results both for the conformal and the non-conformal case.

3.5 The $\mathrm{Sp}'(r)$ gauge group

$\mathrm{Sp}'(r)$ gauge theory is a little mysterious and there is no direct result from the field theory point of view. However, from the brane setup, it is very natural to write down the Seiberg-Witten curve. Loosely speaking, $\mathrm{Sp}'(r)$ with N_f flavors is equivalent to $\mathrm{Sp}(r)$ with $N_f + 1$

[9] For the SO group, we should keep $\Lambda^{2b_{\rm SO}} = qM^{2b_{\rm SO}}$ finite in the large M limit.

flavors, one of which is massless. Using this correspondence, we can heuristically derive the curve for $\mathrm{Sp}'(r)$. First, let us recall the $\mathrm{Sp}(r)$ case, whose curve is given by

$$\prod_{i=1}^{N_f^L}(v^2-m_i^2)t^2+e\left(v^2B(v^2)+8\Lambda^{b_{\mathrm{Sp}}}\prod_{i=1}^{N_f}m_i\right)t+\frac{e^2}{4}(-1)^{b_{\mathrm{Sp}}}64\Lambda^{2b_{\mathrm{Sp}}}\prod_{j=1}^{N_f^R}(v^2-m_j^2)=0\,.$$

To connect with the $\mathrm{Sp}'(r)$ case, we multiply by v both the t^2 term and the t^0 term. This is because the massless half-hypermutiplet corresponds to $\sqrt{v^2-m^2}=v$. Since we have added two half-hypermultiplets, the number of flavors changes from N_f to N_f+1, which implies $b_{\mathrm{Sp}}\to b_{\mathrm{Sp}'}=2r+2-(N_f+1)$. Furthermore,

$$8\Lambda^{2r+2-N_f}\prod_{i=1}^{N_f}m_i\longrightarrow 8\Lambda^{2r+2-N_f-1}\prod_{i=1}^{N_f+1}m_i=0$$

because $m_{N_f+1}=0$. Combining all these facts together we find the curve

$$v\prod_{i=1}^{N_f^L}(v^2-m_i^2)t^2+ev^2B(v^2)t+\frac{e^2}{4}(-1)^{b_{\mathrm{Sp}'}}64\Lambda^{2b_{\mathrm{Sp}'}}v\prod_{j=1}^{N_f^R}(v^2-m_j^2)=0 \qquad (3.8)$$

with $N_f^L+N_f^R=N_f$. We can factor out one v and get

$$\prod_{i=1}^{N_f^L}(v^2-m_i^2)t^2+evB(v^2)t+\frac{e^2}{4}(-1)^{b_{\mathrm{Sp}'}}64\Lambda^{2b_{\mathrm{Sp}'}}\prod_{j=1}^{N_f^R}(v^2-m_j^2)=0\,. \qquad (3.9)$$

In the conformal case, the curve is

$$\prod_{i=1}^{N_f^L}(v^2-m_i^2)t^2+evB(v^2)t+\frac{e^2}{4}g^2(\tau)\prod_{j=1}^{N_f^R}(v^2-m_j^2)=0\,. \qquad (3.10)$$

4. The Splitting of the NS-brane

After this introduction, it is time to test the results on the splitting of NS-branes on $O4$-planes using the Seiberg-Witten curves. Our general framework is depicted in figure 8. We will first write down the Seiberg-Witten curves for three gauge groups, including their dynamical scales, and then check if the curves allow the splitting. We define $v=X^4+iX^5$, $s=(X^6+iX^{10})/R$ (where R is the radius of compact X^{10} direction), and $t=e^{-s}$ as in [21, 29].

4.1 The curve of $\mathrm{Sp}(k_1)\times\mathrm{SO}(2k_2)\times\mathrm{Sp}(k_3)$

The beta-function coefficients are

$$b_{\mathrm{Sp}_1}=2k_1+2-k_2\,,\qquad b_{\mathrm{SO}_2}=2k_2-2-(k_1+k_3)\,,\qquad b_{\mathrm{Sp}_3}=2k_3+2-k_2\,.$$

The curve without dynamical scales is given in [29] by

$$t^4 + \big(v^2 B_1(v^2) + C\big)\, t^3 + B_2(v^2) t^2 + \big(v^2 B_3(v^2) + D\big)\, t + 1 = 0\,.$$

In order to find the explicit dependence on them, we follow the method used in [41]. If we assign the value $2k_1 + 2$ to the dimension of t, then every term in the curve has dimension $8k_1 + 8$. So, we propose the following curve

$$\begin{aligned} t^4 + \big(v^2 B_1(v^2) + C\big)\, t^3 + c_1 \Lambda_1^{2b_{\mathrm{Sp}1}} B_2(v^2) t^2 + \\ + c_2 \Lambda_1^{4b_{\mathrm{Sp}1}} \Lambda_2^{2b_{\mathrm{SO}2}} \big(v^2 B_3(v^2) + D\big)\, t + c_3 \Lambda_1^{6b_{\mathrm{Sp}1}} \Lambda_2^{4b_{\mathrm{SO}2}} \Lambda_3^{2b_{\mathrm{Sp}3}} &= 0\,. \end{aligned} \tag{4.1}$$

The constants C, D are determined by requiring that the curve (4.1) has double roots at $v = 0$ [29]. After some algebra, we find

$$\begin{aligned} C &= \sqrt{4 c_1 \Lambda_1^{2b_{\mathrm{Sp}1}} B_2(0) - 8(x\sqrt{c_3}) \Lambda_1^{3b_{\mathrm{Sp}1}} \Lambda_2^{2b_{\mathrm{SO}2}} \Lambda_3^{b_{\mathrm{Sp}3}}}\,, \\ D &= C \frac{x\sqrt{c_3}}{c_2} \Lambda_1^{-b_{\mathrm{Sp}1}} \Lambda_3^{b_{\mathrm{Sp}3}}\,, \end{aligned} \tag{4.2}$$

where x can be ± 1.

After fixing C and D, we solve for the constants c_1, c_2, c_3 by taking various decoupling limits of (4.1) and matching the result with the curves for single gauge groups. For example, sending first Λ_3 to zero and then Λ_2, we decouple the gauge groups $\mathrm{SO}(2k_2)$ and $\mathrm{Sp}(k_3)$, and we are left with the curve

$$t^2 + (v^2 B_1(v^2) + C') t + c_1 (\Lambda_1^{2b_{\mathrm{Sp}1}}) B_2(v^2) = 0\,.$$

Matching the above with the curve of $\mathrm{Sp}(k_1)$ in table (3.6), we find

$$c_1 = 16(-1)^{b_{\mathrm{Sp}1}}\,.$$

It is a little tricky to determine the constant c_2. Again, we decouple $\mathrm{Sp}(k_3)$ by letting $\Lambda_3 \to 0$. This yields

$$t^3 + \big(v^2 B_1(v^2) + C\big)\, t^2 + c_1 \Lambda_1^{2b_{\mathrm{Sp}1}} B_2(v^2) t + c_2 \Lambda_1^{4b_{\mathrm{Sp}1}} \Lambda_2^{2b_{\mathrm{SO}2}} \big(v^2 B_3(v^2) + D\big) = 0\,.$$

To decouple $\mathrm{Sp}(k_1)$, we need to rescale t first. Substituting t with $t\Lambda_1^{2b_{\mathrm{Sp}1}}$ in the above equation, cancelling the common factor $\Lambda_1^{4b_{\mathrm{Sp}1}}$ and then letting $\Lambda_1 \to 0$, we finally come to

$$\big(v^2 B_1(v^2) + C\big)\, t^2 + c_1 B_2(v^2) t + c_2 (\Lambda_2^{2b_{\mathrm{SO}2}}) \big(v^2 B_3(v^2) + D\big) = 0\,.$$

In this limit, we also have $C = 0$, $D = 0$. Matching with the $\mathrm{SO}(2k_2)$ curve in table (3.6), we find

$$c_2 = 16 c_1^2 (-)^{b_{\mathrm{SO}2}+1} = 16^3 (-)^{b_{\mathrm{SO}2}+1}\,.$$

Repeating the procedure once again, we find

$$\frac{c_3}{c_1} = 16 \left(\frac{c_2}{c_1}\right)^2 (-1)^{b_{\mathrm{Sp}3}}\,, \qquad D' = 8\Lambda_3^{b_{\mathrm{Sp}3}} \sqrt{(-1)^{k_2} B_2(0)}\,.$$

Therefore

$$c_1 = 16(-1)^{b_{\mathrm{Sp}1}}\,, \qquad c_2 = 16^3(-1)^{b_{\mathrm{SO}_2}+1}\,, \qquad c_3 = 16^6(-1)^{b_{\mathrm{Sp}3}} = 16^6(-1)^{b_{\mathrm{Sp}1}+b_{\mathrm{Sp}3}} \tag{4.3}$$

and $x = (-1)^{b_{\mathrm{SO}_2}+1}$.

Note that the curve (4.1) is symmetric under the transformation $t \to t$ and $v \to -v$. This is a consequence of the orientifold action, which reads $X^{6,10} \to X^{6,10}$ and $X^{4,5} \to -X^{4,5}$ and is equivalent to the above transformations of t and v [29].

If $b_{\mathrm{SO}_2} = 0$, the curve is obtained from the above by replacing $16(-1)^{b_{\mathrm{SO}_2}}\Lambda_2^{2b_{\mathrm{SO}_2}}$ with $f(\tau_2)$, where $f(\tau_2)$ is given in (3.3),

$$\begin{aligned} &t^4 + (v^2 B_1(v^2) + C)t^3 + 16(-1)^{b_{sp1}}\Lambda_1^{2b_{\mathrm{Sp}1}} B_2(v^2)t^2 - \\ &- 2^8\Lambda_1^{4b_{\mathrm{Sp}1}} f(\tau_2)(v^2 B_3(v^2) + C\Lambda_1^{-b_{\mathrm{Sp}1}}\Lambda_3^{b_{\mathrm{Sp}3}})t + 2^{16}\Lambda_1^{6b_{\mathrm{Sp}1}} f(\tau_2)^2 \Lambda_3^{2b_{\mathrm{Sp}3}} = 0 \end{aligned} \tag{4.4}$$

and

$$C = 8\Lambda_1^{b_{\mathrm{Sp}1}}\sqrt{(-1)^{b_{\mathrm{Sp}1}} B_2(0) + 2^5\Lambda_1^{b_{\mathrm{Sp}1}} f(\tau_2)\Lambda_3^{b_{\mathrm{Sp}3}}}\,. \tag{4.5}$$

4.2 Lifting of the middle 1/2NS-branes

Our goal is to understand which conditions the curve (4.1) must satisfy in order to be able to lift the middle 1/2NS-branes, B1 and B2, out of the orientifold plane (see part (*c*) of figure 8). The remaining configuration, with A2 and C1 only, would describe a single symplectic gauge group theory.

First of all, if we want to move the two 1/2NS-branes B1 and B2 out of the orientifold plane, we have to reconnect the D4-branes to the left of B1 with the D4-branes to the right of B2. In particular, the two symplectic gauge groups at both ends of the configuration must have the same rank. Thus, we will set $k_1 = k_3$ in (4.1), which leads to $b_{\mathrm{Sp}_1} = b_{\mathrm{Sp}_3} \equiv b_{\mathrm{Sp}}$.

Furthermore, since in this case the left and right gauge theory are the same, we can write the curve in a more symmetric form which is invariant under the exchange of the left and right gauge groups. Recalling that X^6 and X^{10} are the coordinates along the orientifold plane, the above requirement amounts to the curve (4.1) being symmetric under $X^{6,10} \to -X^{6,10}$, after setting the origin of these coordinates appropriately. This transformation is equivalent to $t \to 1/t$.

This gives a relation between $B_1(v^2)$ and $B_3(v^2)$ in (4.1). Setting $\Lambda_1 = \Lambda_3 \equiv \Lambda$, and performing the rescaling $t \to i2^6\Lambda^{2b_{\mathrm{Sp}}}\Lambda_2^{b_{\mathrm{SO}_2}} t \equiv At$, the curve becomes

$$\mathcal{P}(v^2, t) \equiv A^4t^4 + (v^2B_1(v^2) + C)A^3t^3 + c_1\Lambda^{2b_{\mathrm{Sp}}}B_2(v^2)A^2t^2 + (v^2B_3(v^2) + C)A^3t + A^4 = 0\,, \tag{4.6}$$

where we have used the fact that $C = D$ by (4.2). Finally, we see that, in order to achieve the $t \to 1/t$ symmetry,

$$t^4\mathcal{P}\left(v^2, \frac{1}{t}\right) = \mathcal{P}(v^2, t)\,, \tag{4.7}$$

$B_1(v^2)$ must be equal to $B_3(v^2)$.

At a certain stage in the lifting procedure, the middle NS5-branes will coincide. We want to argue that in this limit

$$\mathcal{P}(v^2,t) \to \mathcal{R}(t)\mathcal{Q}(v^2,t)\,, \tag{4.8}$$

where $\mathcal{R}(t)$ is a quadratic polynomial in t with a double root and $\mathcal{Q}(v^2,t)$ is another quadratic polynomial in t with some v dependence describing an $\mathrm{Sp}(r)$ theory with no flavours. Recall from [21] and [29] that the roots in t of $\mathcal{P}(v^2,t)$ at a given v correspond to the positions of the NS5-branes in the t-plane. Requiring no v dependence and a double root for $\mathcal{R}(t)$ precisely depicts the situation where the middle NS5-branes are on top of each other. Furthermore, we can manipulate (3.6) so that $\mathcal{Q}(v^2,t)$ itself will have a $t \to 1/t$ symmetry

$$t^2\mathcal{Q}\left(v^2,\frac{1}{t}\right) = \mathcal{Q}(v^2,t)\,.$$

We are now going to show that for the case at hand, namely when the middle orthogonal theory is not conformal, the splitting procedure is not possible.

First of all, let us find $\mathcal{R}(t)$. Then, equations (4.7) and (4.8) imply that

$$t^4\mathcal{R}\left(\frac{1}{t}\right)\mathcal{R}^{-1}(t) = \mathcal{Q}(v^2,t)\mathcal{Q}\left(v^2,\frac{1}{t}\right)^{-1} = \mathcal{Q}(0,t)\mathcal{Q}\left(0,\frac{1}{t}\right)^{-1} = t^2,$$

which fixes $\mathcal{R}(t)$ to be proportional to $(t+x)^2$, where $x=\pm 1$. Setting

$$\mathcal{P}(v^2,t) = (t+x)^2\left(2^8\Lambda_1^{2b_{\mathrm{Sp}_1}}\Lambda_2^{2b_{\mathrm{SO}_2}}t^2 + \left(e^2v^2\tilde{B}(v^2)+\beta\right)t + 2^8\Lambda_1^{2b_{\mathrm{Sp}_1}}\Lambda_2^{2b_{\mathrm{SO}_2}}\right) = 0\,,$$

we derive the following relations

$$-4i\Lambda_2^{b_{\mathrm{SO}_2}}\left(v^2B_1(v^2)+C\right) = e^2v^2\tilde{B}(v^2)+\beta\,, \tag{4.9}$$

$$(-1)^{b_{\mathrm{Sp}_1}+1}B_2(v^2) = 2x\left(e^2v^2\tilde{B}(v^2)+\beta\right) + 2^9\Lambda_1^{2b_{\mathrm{Sp}_1}}\Lambda_2^{2b_{\mathrm{SO}_2}}\,. \tag{4.10}$$

In particular

$$(-1)^{b_{\mathrm{Sp}_1}+1}B_2(v^2) - 2^9\Lambda_1^{2b_{\mathrm{Sp}_1}}\Lambda_2^{2b_{\mathrm{SO}_2}} = -8xi\Lambda_2^{b_{\mathrm{SO}_2}}\left(v^2B_1(v^2)+C\right),$$

implying that, in the non trivial case, namely when $\Lambda_2^{b_{\mathrm{SO}_2}}$ does not vanish, $v^2B_1(v^2)$ and $B_2(v^2)$ must have the same order. But this is equivalent to $k_1+1=k_2$ or $b_{\mathrm{SO}_2}=0$. Therefore, we conclude that the lifting cannot happen when the middle orthogonal theory is not conformal.

Let us turn to the curve (3.7) describing the conformal case. If we set $f(\tau_2)=-\alpha^2$, with α to be determined, and perform a redefinition of t, $t \to 2^4\Lambda_1^{2b_{\mathrm{Sp}_1}}\alpha\, t$, the curve becomes

$$\begin{aligned}2^{12}\Lambda_1^{6b_{\mathrm{Sp}_1}}\alpha^2\Big(&2^4\alpha^2\Lambda_1^{2b_{\mathrm{Sp}_1}}t^4 + \alpha\left(v^2B_1(v^2)+C\right)t^3 + \\ &+(-1)^{b_{sp1}}B_2(v^2)t^2 + \alpha\left(v^2B_1(v^2)+C\right)t + 2^4\alpha^2\Lambda_1^{2b_{\mathrm{Sp}_1}}\Big) = 0\,.\end{aligned} \tag{4.11}$$

In this form, the curve has a $t \to 1/t$ symmetry. In the limit where the middle NS5-branes coincide, this should be equal to

$$2^{12}\Lambda_1^{6b_{\rm Sp_1}}\alpha^2(t+x)^2\left(2^4\Lambda_1^{2b_{\rm Sp_1}}\alpha^2t^2+\left(e^2v^2\tilde{B}(v^2)+\beta\right)t+2^4\Lambda_1^{2b_{\rm Sp_1}}\alpha^2\right)=0\,,$$

where $x=\pm 1$. This equation yields

$$\alpha\left(v^2B_1(v^2)+C\right)=e^2v^2\tilde{B}(v^2)+\beta+2^5x\alpha^2\Lambda_1^{2b_{\rm Sp_1}} \tag{4.12}$$

and

$$(-1)^{b_{\rm Sp_1}}B_2(v^2)=2xe^2v^2\tilde{B}(v^2)+2x\beta+2^5\alpha^2\Lambda_1^{2b_{\rm Sp_1}}\,. \tag{4.13}$$

Therefore

$$(-1)^{b_{\rm Sp_1}}B_2(v^2)=2x\alpha\left(v^2B_1(v^2)+C\right)-2^5\alpha^2\Lambda_1^{2b_{\rm Sp_1}}\,. \tag{4.14}$$

Since the highest order coefficients both in $B_1(v^2)$ and $B_2(v^2)$ are 1, the above equation implies that $\alpha=\frac{(-1)^{b_{\rm Sp_1}}}{2}x$ and

$$f(\tau_2)=-\frac{1}{4}\,. \tag{4.15}$$

Furthermore, using (4.5), where we set $\Lambda_1^{b_{\rm Sp_1}}=\Lambda_3^{b_{\rm Sp_3}}$, (4.12) and (4.13), we get that

$$\alpha C=2^6x\alpha^2\Lambda_1^{b_{\rm Sp_1}}\,,\quad \beta=2^5x\alpha^2\Lambda_1^{b_{\rm Sp_1}}\,.$$

The remaining symplectic theory is then described by the curve

$$2^4\alpha^2(\Lambda_1^{2b_{\rm Sp_1}})t^2+\alpha\left(v^2B_1(v^2)+2^5x\alpha\Lambda_1^{2b_{\rm Sp_1}}\right)t+2^4\alpha^2(\Lambda_1^{2b_{\rm Sp_1}})=0\,, \tag{4.16}$$

to be compared with

$$t^2+e\left(v^2\tilde{B}(v^2)+8\Lambda^{b_{\rm Sp}}\right)t+16e^2\Lambda^{2b_{sp}}=0.$$

Note that $b_{\rm Sp}=2k_1+2=2b_{\rm Sp_1}=2(2k_1+2-k_2)=2(k_1+1)$. By setting $2^2\alpha\Lambda_1^{b_{\rm Sp_1}}t\to t$, (4.16) becomes

$$t^2+\frac{1}{4}\Lambda_1^{-b_{\rm Sp_1}}\left(v^2B_1(v^2)+2^5x\alpha\Lambda_1^{2b_{\rm Sp_1}}\right)t+2^4\alpha^2(\Lambda_1^{2b_{\rm Sp_1}})=0$$

and everything actually works if we make the following identifications

$$\tilde{B}(v^2)=B_1(v^2)\,,\qquad e=\frac{1}{4}\Lambda_1^{-b_1}\,,\qquad \Lambda^{b_{\rm Sp}}=2(-1)^{b_{\rm Sp_1}}\Lambda_1^{2b_{\rm Sp_1}}\,, \tag{4.17}$$

namely

$$16e^2\Lambda^{2b_{sp}}=2^4\alpha^2\Lambda_1^{2b_{\rm Sp_1}}=4\Lambda_1^{2b_{\rm Sp_1}}\,.$$

In summary, we have seen that we can lift the middle 1/2NS5-branes out of the configuration provided that $b_{\rm SO_2}=(2k_2-2-2k_1)=0$, which is equivalent to

$$k_2=k_1+1\,.$$

If we look up the $O4^+$-plane in table (3.1), we see that this corresponds to $N'=N+1$. Therefore, we reproduced the result listed in table (1.1) that a D4-brane is created when a physical NS5-brane splits on an $O4^+$-plane.

4.3 Consistency check

Note that, as the two middle 1/2NS-branes approach one another, the gauge coupling $g_{\rm YM}$ of the orthogonal theory increases and θ-angle becomes smaller. This is because $1/g^2_{\rm YM}$ and θ are proportional to their relative displacements ΔX^6 and ΔX^{10} respectively [21, 29].

Therefore, in the limit we just considered, $g_{\rm YM}$ should diverge and θ should vanish. This means that equation (4.15) must have $\tau_2 = 0$ as a solution. Using the definition of $f(\tau_2)$ (3.3), we find

$$\vartheta_2^4(\tau_2)\vartheta_4^4(\tau_2) = -\frac{1}{4}\big(\vartheta_2^4(\tau_2) - \vartheta_4^4(\tau_2)\big)^2 \Rightarrow \left(\frac{1}{2}\vartheta_2^4(\tau_2) + \frac{1}{2}\vartheta_4^4(\tau_2)\right)^2 = 0\,,$$

and by the Jacobi identity this is equivalent to

$$\vartheta_3(\tau_2) = 0\,.$$

The zero points of $\vartheta_3(\tau_2)$ are given by

$$0 = \frac{\pi}{2} + \frac{\pi}{2}\tau_2 + m\pi + n\pi\tau_2\,, \quad m, n, \in \mathbb{Z}\,,$$

which do not admit $\tau_2 = 0$ as a solution. However, as we observed above and we explain in appendix B, we could select $f(\tau_2 + 1)$ as our modular function, leading to physically equivalent results. The condition for the splitting would then become $\vartheta_3(\tau_2+1) = \vartheta_4(\tau_2) = 0$, equivalently

$$0 = \frac{\pi}{2}\tau_2 + m'\pi + n'\pi\tau_2\,, \quad m', n', \in \mathbb{Z}\,,$$

which admits $\tau_2 = 0$ as a solution.

4.4 The curve of $\mathrm{SO}(2k_1) \times \mathrm{Sp}(k_2) \times \mathrm{SO}(2k_3)$

The beta-function coefficients of the three groups are

$$b_{\mathrm{SO}_1} = 2k_1 - 2 - k_2\,, \qquad b_{\mathrm{Sp}_2} = 2k_2 + 2 - (k_1 + k_3)\,, \qquad b_{\mathrm{SO}_3} = 2k_3 - 2 - k_2\,.$$

The curve without dynamical scales in [29] is

$$v^2t^4 + B_1(v^2)t^3 + \big(v^2B_2(v^2) + C\big)\,t^2 + B_3(v^2)t + v^2 = 0$$

Again, if we assign the value $2k_1 - 2$ to the dimension of t, the curve will be given by

$$\begin{aligned} v^2t^4 + B_1(v^2)t^3 + c_1(\Lambda_1^{2b_{\mathrm{SO}_1}})\,\big(v^2B_2(v^2) + C\big)\,t^2 + \\ + c_2(\Lambda_1^{2b_{\mathrm{SO}_1}})^2(\Lambda_2^{2b_{\mathrm{Sp}_2}})B_3(v^2)t + c_3(\Lambda_1^{2b_{\mathrm{SO}_1}})^3(\Lambda_2^{2b_{\mathrm{Sp}_2}})^2(\Lambda_3^{2b_{\mathrm{SO}_3}})v^2 = 0 \end{aligned} \qquad (4.18)$$

The constant C is determined by requiring that the above equation has a double root at $v = 0$. Therefore

$$C = \pm 2\frac{\Lambda_2^{b_{\mathrm{Sp}_2}}}{c_1}\sqrt{c_2B_1(0)B_3(0)}\,.$$

By decoupling the various groups, we find the coefficients to be

$$c_1 = 16(-1)^{b_{\mathrm{SO}_1}+1}\,, \qquad c_2 = 16^3(-1)^{b_{\mathrm{Sp}_2}}\,, \qquad c_3 = 16^6(-1)^{b_{\mathrm{SO}_1}+1+b_{\mathrm{SO}_3}+1} = 16^6\,. \quad (4.19)$$

Finally, let us consider the conformal case, $b_{\mathrm{Sp}_2}=0$, where $2k_2+2=k_1+k_3$. In this case, we replace $8\Lambda_2^{2k_2+2-k_1-k_3}$ with $g(\tau_2)$, as in (3.3), and rewrite (4.18) as

$$\begin{aligned} & v^2t^4+B_1(v^2)t^3+c_1(\Lambda_1^{2b_{\mathrm{SO}_1}})(v^2B_2(v^2)+C)t^2+ \\ & +c_2(\Lambda_1^{2b_{\mathrm{SO}_1}})^2g^2(\tau_2)B_3(v^2)t+c_3(\Lambda_1^{2b_{\mathrm{SO}_1}})^3g^4(\tau_2)(\Lambda_3^{2b_{\mathrm{SO}_3}})v^2=0 \end{aligned} \tag{4.20}$$

where

$$c_1=16(-1)^{b_{\mathrm{SO}_1}+1}, \qquad c_2=64, \qquad c_3=16^3, \tag{4.21}$$

and

$$C=g(\tau_2)\sqrt{(-1)^{k_1+k_3}B_1(v^2=0)B_3(v^2=0)}. \tag{4.22}$$

We then proceed to discuss the lifting of the middle 1/2NS-branes. Setting $k_1=k_3$, $B_1(v^2)=B_3(v^2)$, $\Lambda_1^{2b_{\mathrm{SO}_1}}=\Lambda_3^{2b_{\mathrm{SO}_3}}$, and performing the rescaling $t\to 2^6\Lambda_1^{2b_{\mathrm{SO}_1}}\Lambda_2^{b_{\mathrm{Sp}2}}it\equiv At$, the curve (4.18) becomes

$$\mathcal{P}(v^2,t)\equiv A^4v^2t^4+B_1(v^2)A^3t^3+c_1\Lambda_1^{2b_{\mathrm{SO}_1}}\left(v^2B_2(v^2)+C\right)A^2t^2-B_1(v^2)A^3t+A^4v^2=0. \tag{4.23}$$

The curve $\mathcal{P}(v^2,t)$ has a $t\to -1/t$ symmetry

$$t^4\mathcal{P}\left(v^2,-\frac{1}{t}\right)=\mathcal{P}(v^2,t).$$

In the limit where the middle NS5-branes are lifted, $\mathcal{P}(v^2,t)$ reduces to

$$\mathcal{P}(v^2,t)=\mathcal{R}(t)\mathcal{Q}(v^2,t),$$

where the curve $\mathcal{Q}(v^2,t)$ describes an SO$(2r)$ curve with no flavours. We have that

$$t^4\mathcal{P}\left(v^2,-\frac{1}{t}\right)=t^4\mathcal{R}\left(-\frac{1}{t}\right)\mathcal{Q}\left(v^2,-\frac{1}{t}\right)=\mathcal{P}(v^2,t)=\mathcal{R}(t)\mathcal{Q}(v^2,t)$$

is solved by $\mathcal{R}(t)=(t+xi)^2$, where $x=\pm1$. Therefore, setting

$$\mathcal{P}(v^2,t)=(t+xi)^2\left(2^8\Lambda_1^{2b_{\mathrm{SO}_1}}\Lambda_2^{2b_{\mathrm{Sp}2}}v^2t^2+\alpha\tilde{B}(v^2)t-2^8\Lambda_1^{2b_{\mathrm{SO}_1}}\Lambda_2^{2b_{\mathrm{Sp}2}}v^2\right)$$

implies the following identities

$$\begin{aligned} -4i\Lambda_2^{b_{\mathrm{Sp}2}}B_1(v^2) &= \alpha\tilde{B}(v^2)+2^9xi\Lambda_1^{2b_{\mathrm{SO}_1}}\Lambda_2^{2b_{\mathrm{Sp}2}}v^2, && (4.24)\\ (-1)^{b_{\mathrm{SO}_1}}\left(v^2B_2(v^2)+C\right) &= -2^9\Lambda_1^{2b_{\mathrm{SO}_1}}\Lambda_2^{2b_{\mathrm{Sp}2}}v^2+2xi\alpha\tilde{B}(v^2). && (4.25) \end{aligned}$$

In particular we get

$$(-1)^{b_{\mathrm{SO}_1}}\left(v^2B_2(v^2)+C\right)+2^9\Lambda_1^{2b_{\mathrm{SO}_1}}\Lambda_2^{2b_{\mathrm{Sp}2}}v^2=8x\Lambda_2^{b_{\mathrm{Sp}2}}B_1(v^2)+2^{10}\Lambda_1^{2b_{\mathrm{SO}_1}}\Lambda_2^{2b_{\mathrm{Sp}2}}v^2.$$

In the non trivial case, $\Lambda_2^{b_{\mathrm{Sp}2}}\neq 0$, the order of the two polynomials on either side of the equation is the same and therefore $b_{\mathrm{Sp}_2}=2k_2+2-2k_1=0$. Hence, it is not possible to lift the middle NS5-branes when the middle symplectic theory is not conformal.

In the conformal case, we obtain

$$\mathcal{P}(v^2,t) = \Big(2^3\Lambda_1^{2b_{\mathrm{SO}_1}} g^2 v^2 t^4 - igB_1(v^2)t^3 + \\ + 2(-1)^{b_{\mathrm{SO}_1}} \left(v^2 B_2(v^2) + C\right) t^2 igB_1(v^2)t + 2^3\Lambda_1^{2b_{\mathrm{SO}_1}} g^2 v^2\Big) = 0\,.$$

Setting

$$\mathcal{P}(v^2,t) = (t+xi)^2\left(2^3\Lambda_1^{2b_{\mathrm{SO}_1}} g^2 + \alpha\tilde{B}(v^2)t - 2^3\Lambda_1^{2b_{\mathrm{SO}_1}} g^2 v^2\right)$$

we find that

$$-igB_1(v^2) = \alpha\tilde{B}(v^2) + 2^4 xi\Lambda_1^{2b_{\mathrm{SO}_1}} g^2 v^2$$

and

$$2(-1)^{b_{\mathrm{SO}_1}}\left(v^2 B_2(v^2) + C\right) = -2^4\Lambda_1^{2b_{\mathrm{SO}_1}} g^2 v^2 + 2xi\alpha\tilde{B}(v^2)\,.$$

Therefore

$$2(-1)^{b_{\mathrm{SO}_1}}\left(v^2 B_2(v^2) + C\right) + 2^4\Lambda_1^{2b_{\mathrm{SO}_1}} g^2 v^2 = 2xgB_1(v^2) + 2^5\Lambda_1^{2b_{\mathrm{SO}_1}} g^2 v^2\,,$$

which implies that

$$g = (-1)^{b_{\mathrm{SO}_1}} x$$

and

$$C = B_1(0)\,.$$

If we set $C = gB_1(0)$ in (4.22), then $g = 1$ and $x = (-1)^{b_{\mathrm{SO}_1}}$.

In any case, we have $g^2(\tau_2) = 1$, which, by the Jacobi identity, is equivalent to $\vartheta_3^4(\tau)\vartheta_4^4(\tau) = 0$. As we saw in the previous section, $\tau_2 = 0$ is a solution of the above equation, which provides a consistency check on procedure.

In summary, we have seen that we can lift the middle 1/2NS5-branes out of the configuration provided that $b_{\mathrm{Sp}_2} = 2k_2 + 2 - 2k_1 = 0$, which is equivalent to

$$k_2 = k_1 - 1\,.$$

If we look up the $O4^-$-plane in table (3.1), we see that this corresponds to $N' = N - 1$. Therefore, this confirms the result listed in table (1.1) that a D4-brane is annihilated when a physical NS5-brane splits on an $O4^-$-plane.

4.5 The splitting on $\widetilde{O4}^{\pm}$-planes

As we have argued in section 2.3.2, a physical NS-brane splits on an $\widetilde{O4}^{+}$-plane with neither creation nor annihilation of D4-branes and likewise it can split on an $\widetilde{O4}^{-}$-plane with the annihilation of a D4-brane. In order to obtain this result, it was necessary to take into account the energy cost due to the bending of the 1/2NS-branes.

In particular, the minimum energy configuration after the splitting corresponds to the 1/2NS-branes intersecting each other at some distance from the $O4$-plane. Note that this system does not correspond to an asymptotically free theory and therefore we do not expect to see any splitting in the range of validity of the Seiberg-Witten description. This is what we are about to show. In other words, we will show that another choice in table (2.2), which correspond to asymptotically free theory (i.e., 1 for $\widetilde{O4}^{+}$ and 0 for $\widetilde{O4}^{-}$), can not be unsplitted and lifed away from the orientifold planes.

For simplicity, we present only the case of $\mathrm{SO}(2k_1+1)\times\mathrm{Sp}'(k_2)\times\mathrm{SO}(2k_3+1)$. The beta-function coefficients are

$$b_{\mathrm{SO}_1}=(2k_1+1)-2-k_2\,,\qquad b_{\mathrm{Sp}_2}=2k_2+2-(k_1+k_3+1)\,,\qquad b_{\mathrm{SO}_3}=(2k_3+1)-2-k_2\,, \tag{4.26}$$

Both $\mathrm{SO}(2k_1+1)$ and $\mathrm{SO}(2k_3+1)$ have $2k_2$ half-hypers or k_2 hypers. On the other hand, $\mathrm{Sp}'(k_2)$ has $(2k_1+1)+(2k_3+1)$ half-hypers, for a total k_1+k_3+1 hypers. The curve for this gauge theory is given by equation (3.42) in [29]

$$vt^4+B_1(v^2)t^3+vB_2(v^2)t^2+B_3(v^2)t+v=0\,.$$

Assigning the value $2k_1-1$ to the dimension of t, we find the curve to be

$$vt^4+B_1(v^2)t^3+c_1\Lambda_1^{2b_{\mathrm{SO}_1}}vB_2(v^2)t^2+c_2\Lambda_1^{4b_{\mathrm{SO}_1}}\Lambda_2^{2b_{\mathrm{Sp}_2}}B_3(v^2)t+c_3\Lambda_1^{6b_{\mathrm{SO}_1}}\Lambda_2^{4b_{\mathrm{Sp}_2}}\Lambda_3^{2b_{\mathrm{SO}_3}}v=0 \tag{4.27}$$

and taking the various decoupling limits, we determine the coefficients to be

$$c_1=16(-1)^{b_{\mathrm{SO}_1}+1}\,,\qquad c_2=16^3(-1)^{b_{\mathrm{Sp}_2}}\,,\qquad c_3=16^6\,. \tag{4.28}$$

Unlike the previous subsections, the curve (4.27) is invariant under $v\to -v, t\to -t$. Equivalently, the orientifold action is $X^{4,5}\to -X^{4,5}$, $X^6\to X^6$ and $X^{10}\to X^{10}+\pi R$. The symmetrized form of (4.27) is given by rescaling $t\to At\equiv 2^6 i\Lambda^{2b_{\mathrm{SO}}}\Lambda_2^{b_{\mathrm{Sp}2}}$ under the contraints $k_1=k_3$ and $\Lambda_1=\Lambda_3=\Lambda$. We find

$$\mathcal{P}(v^2,t)=vA^4t^4+B_1(v^2)A^3t^3-c_1\Lambda_1^{2b_{\mathrm{SO}_1}}vB_2(v^2)A^2t^2+B_3(v^2)A^3t+vA^4=0\,. \tag{4.29}$$

The curve $\mathcal{P}(v^2,t)$ has a $t\to 1/t$ symmetry

$$t^4\mathcal{P}\left(v^2,\frac{1}{t}\right)=\mathcal{P}(v^2,t)\,.$$

In the limit where the middle NS5-branes are lifted $\mathcal{P}(v^2,t)$, reduces to

$$\mathcal{P}(v^2,t)=\mathcal{R}(t)\mathcal{Q}(v^2,t)\,,$$

where the curve $\mathcal{Q}(v^2,t)$ describes an $\mathrm{SO}(2r+1)$ curve with no flavours.

We have that

$$t^4\mathcal{P}\left(v^2,\frac{1}{t}\right)=t^4\mathcal{R}\left(\frac{1}{t}\right)\mathcal{Q}\left(v^2,\frac{1}{t}\right)=\mathcal{P}(v^2,t)=\mathcal{R}(t)\mathcal{Q}(v^2,t)$$

is solved by $\mathcal{R}(t)=(t+x)^2$, where $x=\pm1$. Therefore, setting

$$\mathcal{P}(v^2,t)=(t+x)^2\left(2^8\Lambda_1^{2b_{\mathrm{SO}_1}}\Lambda_2^{2b_{\mathrm{Sp}2}}vt^2+\alpha\tilde{B}(v^2)t+2^8\Lambda_1^{2b_{\mathrm{SO}_1}}\Lambda_2^{2b_{\mathrm{Sp}2}}v\right)$$

implies the following identities

$$-4i\Lambda_2^{b_{\mathrm{Sp}2}}B_1(v^2)\;=\;\alpha\tilde{B}(v^2)+2^9x\Lambda_1^{2b_{\mathrm{SO}_1}}\Lambda_2^{2b_{\mathrm{Sp}2}}v\,, \tag{4.30}$$

$$(-1)^{b_{\mathrm{SO}_1}+1}vB_2(v^2)\;=\;2^9\Lambda_1^{2b_{\mathrm{SO}_1}}\Lambda_2^{2b_{\mathrm{Sp}2}}v+2x\alpha\tilde{B}(v^2)\,. \tag{4.31}$$

In particular we get

$$(-1)^{b_{\mathrm{SO}_1}+1} v B_2(v^2) - 2^9 \Lambda_1^{2b_{\mathrm{SO}_1}} \Lambda_2^{2b_{\mathrm{Sp}_2}} v = -8xi\Lambda_2^{b_{\mathrm{Sp}_2}} B_1(v^2) - 2^{10} \Lambda_1^{2b_{\mathrm{SO}_1}} \Lambda_2^{2b_{\mathrm{Sp}_2}} v\,.$$

We see that the above equation cannot be satisfied since the two polynomials have different degrees. On the l.h.s. we have a polynomial of odd degree in v, whereas on the r.h.s. the degree is even. Therefore, we conclude that for $k_1 - 1/2 \leq k_2 \leq 2k_1 - 1$, it is not possible to lift the middle 1/2NS5-branes.

As we remarked at the beginning, this result tells us that another choice in table (2.2) is not correct and leaves us with the one given in table (1.1).

5. Conclusion

In this paper, we have studied the general splitting process of NS-branes and D(p+2)-branes on Op-planes with $p \leq 6$. Using several arguments, we showed that for D($p+2$)-brane splitting, the rule is universal, while for NS-brane splitting, the rule is not universal and depends on the dimension p. The results are summarized in table (1.1) in the introduction.

In a special case, namely for NS-branes on $O4$-planes, we can study the process with the appropriate Seiberg-Witten curves. If the splitting is allowed, the reverse process (unsplitting) must also take place. This corresponds to a particular strong coupling limit on the field theory side and was checked explicitly using the curves.

There are a lot of directions we can follow. First, it would still be very nice if we could derive these results directly, for example, from a world-sheet CFT calculation. Second, we could generalize the above discussion to M-theory and consider, for example, the splitting of M5-branes on OM2-planes. There are also other orientifold-like planes, for example $ON5$-planes, which are S-dual to $O5$-planes, and $OF1_{A,B}$-planes. Third, in [6, 7] the mirror pairs of Sp(k) groups are derived by using the $O5^-$ and its S-duality counterparts ON^- and ON^0 (the bound state of ON^- and one physical NS-brane. See [42], and some applications in [43, 44]). It would be natural to use the $O5^+$-plane to study the mirror of SO(n) gauge groups. The problem with this approach is that we do not know the action of S-duality on the ON^+-plane. Fourth, we have predicted the existence of a 1/4D5-brane when NS-branes split on $\widetilde{O5}^{\pm}$-planes and argued that it does not cause any inconsistency. However, further study is still very welcome to clarify this point. Finally, it would be very interesting to find some applications of the above non-trivial string dynamical process.

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A. The generalized supersymmetric configuration

We have discussed the supersymmetric configurations of NS-branes and D5-branes in the case of $O3$-planes. Now we want to see if these results can be generalized to other Op-planes. We will give two different arguments.

The first one relies on the linking number. Let us focus on the supersymmetric configuration containing $O4$-planes first. In this case, $O4^+, \widetilde{O4^+}$ have charge 1/2, $O4^-$ has charge $-1/2$ and $\widetilde{O4^-}$ has charge 0. Using the linking number

$$L_{1/2NS} = \frac{1}{2}(R_{D6} - L_{D6}) + (L_{D4} - R_{D4})$$

as in [8] we get the following constraints on the number of Dp-branes between a 1/2NS-brane and a 1/2D5-brane

$$\begin{array}{ll} (O4^- - - - - - - \widetilde{O4^+}) \ \& \ (\widetilde{O4^+} - - - - - - O4^-) & N + \widetilde{N} = 0 \\ (O4^+ - - - - - - \widetilde{O4^-}) \ \& \ (\widetilde{O4^-} - - - - - - O4^+) & N + \widetilde{N} = 1 \end{array} \tag{A.1}$$

Let us explain the above notation. For example, by $(O4^- - - - - - \widetilde{O4^+})$ we mean the following two brane setups: going from left to right we have either $(O4^-, 1/2NS\text{-brane}, O4^+ + N$ Dp-branes$, 1/2\text{D}(p+2)\text{-brane}, \widetilde{O4^+})$ or $(O4^-, 1/2\text{D}(p+2)\text{-brane}, \widetilde{O4^-} + \widetilde{N}$ Dp-branes, $1/2NS\text{-brane}, \widetilde{O4^+})$. These two are related by shifting the 1/2NS-brane and the 1/2D$(p+2)$-brane. The above brane setup is supersymmetric if and only if $N \geq 0, \widetilde{N} \geq 0$ and they satisfy the equation (A.1). Note that the answer is same as that for the $O3$-planes. In fact, it can be checked by an explicit calculation that the above conclusion is the same for all $p \leq 5$.

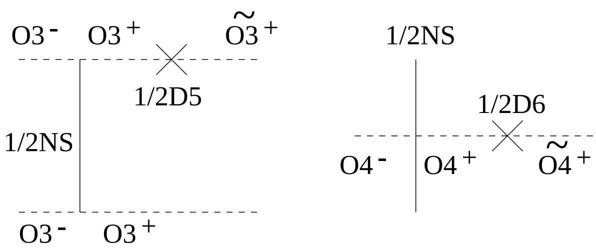

Figure 9: The left-hand side is the correct supersymmetric configuration of $O3$-planes. After T-duality, we get the supersymmetric configuration for $O4$-planes at the right-hand side.

It is possible to retrieve this general result by a T-duality argument. Let us consider $O4$-planes again. Starting from the $O3$-planes we perform a T-duality along the X^3 coordinate (the world volume coordinates of the various branes are: $O3$-plane X^{126}, NS-brane X^{12345} and D5-brane X^{12789}). We draw the brane setup in the (X^6, X^3) plane (see the left-hand side of figure 9). Notice that in this brane setup the 1/2NS-brane intersects two orientifold planes, whereas the 1/2D5-brane intersects only one orientifold plane. After T-duality along X^3, two $O3^-$-planes combine to give one $O4^-$-plane [19], two $O3^+$-planes combine to give one $O4^+$-plane, while one $O3^+$-plane and one $\widetilde{O3^+}$-plane combine to give one $\widetilde{O4^+}$-plane. Therefore, we get the right-hand side of figure 9. The crucial thing is that under T-duality, the number of allowed Dp-branes between the 1/2NS-brane and the 1/2D$(p+2)$-brane is invariant. This explain why we have the above general result.

B. Ambiguity in the modular function

The curves describing the Coulomb branch of SU$(r+1)$, Sp(r) and SO$(2r)$ theories in the conformal case contain some modular functions of $\tau = \frac{\theta}{\pi} + i\frac{8\pi}{g^2}$. These expressions are determined recursively in the rank of the group and by matching the Seiberg-Witten curve for SU(2) [35, 39].

We want to explain why there are actually two physically equivalent expressions, which differ by $\tau \to \tau + 1$.

The SU$(r+1)$ curve with $N_f = 2r+2$ reads [39]

$$y^2 = \prod_{a=1}^{r+1} (x - \phi_a)^2 + 4h(h+1) \prod_{j=1}^{2r+2} (x - m_j - 2h\mu)\,,$$

where

$$h(\tau) = \frac{\vartheta_2^4}{\vartheta_4^4 - \vartheta_2^4}\,. \tag{B.1}$$

For SO$(2r+1)$ and SO$(2r)$, the modular function $f(\tau)$ in (3.2) is expressed in terms of $h(\tau)$ [39]. This is achieved by matching the SO(3) curve with one vector hypermultiplet to the SU(2) curve with a massless adjoint hypermultiplet.

In particular

$$f(\tau) = h(\tau)(h(\tau) + 1)\,. \tag{B.2}$$

In an earlier paper [35], the authors analyzed the curves for $\mathcal{N} = 2$ SU$(r+1)$ gauge theories and showed that for the conformal curve you may choose two different modular functions which are related to one another by $\tau \to \tau + 1$. Both choices have the correct weak-coupling asymptotic behaviour and match the Seiberg-Witten curve for SU(2). This amounts to a convential choice of the origin of the θ angle. This implies that, in (B.2), we can take either $h(\tau)$ or $h(\tau+1)$. Therefore, $f(\tau)$ or $f(\tau+1)$ are physically indistinguishable.

For Sp(r), the unknown modular function $g(\tau)$ is recovered by matching to the Seiberg-Witten solution for SU(2) $\cong$ Sp(1). The map $\tau \to \tau + 1$ is a generator of the duality group [39], under which g goes to $-g$. Then the symplectic curve in (3.2) is invariant if the sign of a single mass is changed.

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